Kuwait University | 'Math 102 | Date: 02/07/2005

Department of Math & CS | First Test | Duration: 75 min.

Calculators, Pagers, and Mobile phones are not allowed.

Instructions: (i) Read each question carefully. (ii) Show all work in order to receive full credit.

1. Let $f(x) = \ln(x^2+2)^2, \qquad x<0$
   - (a) Prove that f has an inverse (2 pts.)
   - (b) Find f^{-1} and its domain. (3 pts.)

2. Find dy/dx
   - (a) $y = \sqrt{\dfrac{x^2(2x-1)^3}{(x+5)^2}}$ (2 pts.)
   - (b) $e^{xy} = \tan^{-1}(\sinh y) + (1+x)^x$ (3 pts.)

3. Answer the following questions
   - (a) Simplify

$$\cosh\left(\frac{1}{2}\ln(x+1) - \ln 2\right)$$

   and find its value in a rational form. (2pts.)

   - (b) Show that (2pts.)

$$\cot^{-1}\left(\frac{\sqrt{1-x}}{\sqrt{x}}\right) = \sin^{-1}(\sqrt{x})$$

4. Evaluate the following integrals:
   - (a) $\displaystyle\int \frac{dx}{5x^{\frac{1}{2}} + x^{\frac{3}{2}}}$ (2 pts.)
   - (b) $\displaystyle\int \tan[\ln(x^x e^x)] \ln\left(\frac{1}{xe^2}\right) dx$ (3 pts.)

5. Evaluate the limit, if it exists:
   - (a) $\displaystyle\lim_{x\to\infty} (e^x)^{e^{-x}}$ (3 pts.)
   - (b) $\displaystyle\lim_{x\to 1} \frac{\ln x^{\sqrt{x}}}{x^3-1}$ (3 pts.)

Math 102 Kuwait University

First Test (Key), dated: 02/07/2005 Departement of Math & CS

1. (a) Let $y = 2\ln(x^2+2)$, then $\frac{dy}{dx} = \frac{4x}{x^2+2} < 0$. So, $f(x)$ is an ~~increasing~~ [decreasing] function $\forall x \in$ ~~$(0,\infty)$~~ $(-\infty, 0)$. Hence, f is 1-1 and f^{-1} exists.

(b) $\frac{y}{2} = \ln(x^2+2) \Rightarrow e^{\frac{y}{2}} = x^2+2 \Rightarrow x^2 = e^{\frac{y}{2}} - 2 \Rightarrow x = -\sqrt{e^{y/2}-2}$ for ~~$x>0$~~ $X<0$.

$$\boxed{\text{Hence } f^{-1}(x) = -\sqrt{e^{x/2}-2}.\ \text{Domain } f^{-1} \text{ is } [2\ln 2, \infty)}.$$

2. (a) $\ln y = \frac{1}{2}\ln\frac{x^2(2x-1)^3}{(x+5)^2} = \frac{1}{2}[2\ln x + 3\ln(2x-1) - 2\ln(x+5)]$. Differentiating w.r.t. x:

$$\Rightarrow \frac{y'}{y} = \frac{1}{2}\left[\frac{2}{x} + \frac{6}{2x-1} - \frac{2}{x+5}\right] \Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x} + \frac{3}{2x-1} - \frac{1}{x+5}\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \ln\sqrt{\frac{x^2(2x-1)^3}{(x+5)^2}}\left[\frac{3x^2+25x-5}{x(2x-1)(x+5)}\right]}$$

(b) Let $\frac{d}{dx}(1+x)^x = \frac{d}{dx}e^{x\ln(1+x)} = e^{x\ln(1+x)}\left[\ln(1+x) + \frac{x}{1+x}\right] = (1+x)^x\left[\ln(1+x) + \frac{x}{1+x}\right]$. So, differentiating all terms implicitly w.r.t. x:

$$e^{xy}\left(y + xy'\right) = \frac{y'\cosh y}{1+\sinh^2 y} + (1+x)^x\left[\ln(1+x) + \frac{x}{1+x}\right] \Rightarrow$$

$$y'e^{xy} - y'\operatorname{sech} y = (1+x)^{x-1}\left[\ln(1+x)^{(1+x)} + x\right] - ye^{xy} \Rightarrow$$

$$\boxed{y' = \frac{(1+x)^{x-1}\left[\ln(1+x)^{(1+x)} + x\right] - ye^{xy}}{e^{xy} - \operatorname{sech} y}}$$

3. (a) $\cosh\left(\frac{1}{2}\ln(x+1) - \ln 2\right) = \cosh\left(\ln\frac{(x+1)^{1/2}}{2}\right) = \frac{e^{\ln\frac{(x+1)^{1/2}}{2}} + e^{-\ln\frac{(x+1)^{1/2}}{2}}}{2}$, simplifying

$$\Rightarrow \frac{e^{\ln\frac{(x+1)^{1/2}}{2}} + e^{\ln\left[\frac{(x+1)^{1/2}}{2}\right]^{-1}}}{2} = \frac{\frac{(x+1)^{1/2}}{2} + \frac{2}{(x+1)^{1/2}}}{2} = \boxed{\frac{x+5}{4(x+1)^{1/2}}}.$$

(b) Let $\theta = \cot^{-1}\left(\frac{\sqrt{1-x}}{\sqrt{x}}\right)$, then $\cot\theta = \frac{\sqrt{1-x}}{\sqrt{x}}$. So, $\sin\theta = \sqrt{x} \Rightarrow \sin^{-1}(\sqrt{x})$ is true.

4. (a) $\int\frac{dx}{5x^{1/2}+x^{3/2}} = \int\frac{dx}{x^{1/2}(5+x)} \Rightarrow$ Let $u = x^{1/2}$, then $du = \frac{dx}{2x^{1/2}}$

$$\Rightarrow 2\int\frac{du}{5+u^2} = \boxed{\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{x}{5}} + C}.$$

(b) $\int\tan[\ln(x^x e^x)]\ln\left(\frac{1}{xe^2}\right)dx = -\int\tan[\ln(x^x e^x)]\ln(xe^2)\,dx \Rightarrow$ Let $u = \ln(x^x e^x) = x\ln x + x$, then $du = (\ln x + 2)dx = \ln(xe^2)dx \Rightarrow -\int\tan u\,du = \boxed{-\ln|\sec[\ln(x^x e^x)]| + C}$.

5. (a) $\lim_{x\to\infty}(e^x)^{e^{-x}}$ has the Indeterminate form ∞^0. Let $y = (e^x)^{e^{-x}} \Rightarrow \ln y = e^{-x}\ln e^x = xe^{-x}$. Then $\lim_{x\to\infty}\ln y = \lim_{x\to\infty}xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x}$ has the form $\frac{\infty}{\infty}$. Apply L'H. rule $\Rightarrow \lim_{x\to\infty}\frac{1}{e^x} = 0$. So, $\lim_{x\to\infty}\ln y = 0 \Rightarrow \lim_{x\to\infty}y = e^0 = \boxed{1}$.

(b) $\lim_{x\to 1}\frac{\ln x^{\sqrt{x}}}{x^3-1} = \lim_{x\to 1}\frac{\sqrt{x}\ln x}{x^3-1}$ has the Indeterminate form $\frac{0}{0}$. Apply L'H. rule

$$\Rightarrow \lim_{x\to 1}\frac{\frac{\sqrt{x}}{x} + \frac{\ln x}{\sqrt{x}}}{3x^2} = \boxed{\frac{1}{3}}$$